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Scaling Lee-Yang model on a sphere 1. Partition function*

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Abstract: Some general properties of perturbed (rational) CFT in the background metric of symmetric 2D sphere of radius R are discussed, including conformal perturbation theory for the partition function and the large R asymptotic. The truncated conformal space scheme is adapted to treat numerically perturbed rational CFT's in the spherical background. Numerical results obtained for the scaling Lee-Yang model lead to the conclusion that the partition function is an entire function of the coupling constant. Exploiting this analytic structure we are able to describe, starting only with the CFT information and few first terms of conformal perturbation theory, rather precisely the "experimental" truncated space data, including even the large R behavior.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Conformal and W Symmetry, 2D Gravity.

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1. Motivations

There are several reasons to be interested in euclidean 2D field theory in spherical background. Apart from conceptual interest in relativistic field theory on curved spaces, the particular example of sphere presents an apparent simplification of a symmetric space where physics is independent on the space position while being different from that on the infinite flat plane.

It has been proven useful to study properties of interacting field theory through finite-size effects (see e.g. [1, 2, 3]). The compact geometry of sphere naturally provides a settlement of finite size problem, different from the Casimir effect which has been mainly studied so far.

Exact constructions of various conformal field theories (CFT) in 2D gave rise to a very fruitful idea to consider non-conformal (both massive and massless) field theory models as certain perturbations of CFT's by relevant operators [4]. One considers formal actions like

$$\mathcal{A}_\lambda = \mathcal{A}_{\text{CFT}} + \frac{\lambda}{2\pi}\int \phi(x) d^2x \tag{1.1}$$

where ϕ is some relevant primary field (of dimension Δ) in the theory and λ is the corresponding coupling constant. Even if such perturbation leads to a field theory which is not integrable, the perturbation theory in λ, called the conformal perturbation theory (CPT), may give valuable information about the short distance behavior of the model. One of the main problems of CPT is related to infrared divergences of the perturbative integrals, which result in certain non-analyticity in λ. Finite geometry of a sphere imposes an infrared cutoff. It is natural to expect that on a finite sphere observables are analytic in λ at $\lambda = 0$. Below I'll try to argue that in certain cases the CPT series on a finite sphere is *absolutely convergent* and therefore results in entire function of λ. Although there are no obvious reasons to expect any sort of integrability on the sphere even if the model (1.1) is integrable on the flat plane, the analytic property suggested is extremely restrictive. As we'll see before long, it permits to get a lot of information starting from few first CPT coefficients and some natural assumptions about $\lambda \to \infty$ asymptotic.

There is an approximate way to treat the actions like (1.1) called the truncated conformal space (TCS) approach [3]. Sometimes it gives rather precise numerical information even about the large-distance properties of the field theory [5]. Spherical geometry seems to be a very natural environment for the TCS approach.

The main motivation, at least for me, to study perturbed CFT's on fixed sphere is the long standing challenge of 2D quantum gravity. There are all reasons to believe that the field theoretic approach based on the Liouville field theory is a relevant way to understand 2D gravity at least in the so-called "week coupling regime". However, up to now only very basic information about general scale dimensions and simplest correlation functions is reached by the field theoretic means, even in the simplest case of spherical topology. This has to be compared with very reach "experimental" results coming from the matrix models and similar "discrete geometry" approaches (see e.g., the review [6] or ref. [7] for most recent achievements in the field). I hope that better understanding of the field theory on a fixed symmetric sphere can be a starting point to approach the problem at least in the semiclassical limit where the central charge of the massless matter modes is large negative and the geometry becomes "rigid".

2. CFT in curved background

The most advanced understanding of the 2D field theory in curved background $g_{ab}(x)$ is reached in conformal field theory. This is due to the following two simple properties postulated as a basis of CFT construction

Stress tensor anomaly. In 2D CFT the trace of the stress tensor $T_{ab}(x)$ is in fact a c-number (i.e., proportional to the identity operator) and reads explicitly

$$\theta(x) = g^{ab}T_{ab} = -\frac{c}{12}\widehat{R} \tag{2.1}$$

where $\widehat{R}$ is the scalar curvature of the background metric and c is certain (real) number, called the central charge. The central charge is the basic characteristic for any particular CFT.

Primary fields. Among the set of local fields $\{\Phi\}$ in the theory there is a subset of so called primary fields ϕ_i which transform very simply (here the primary fields are supposed to be scalars)

$$\delta\phi_i(x) = -\Delta_i\phi_i(x)\delta\varphi(x) \tag{2.2}$$

under the Weil variations of the metric

$$\delta g_{ab}(x) = g_{ab}(x)\delta\varphi(x) \tag{2.3}$$

In (2.2) Δ_i are (real) numbers characteristic for the primary fields, called the scale dimensions (or conformal weights). All other local fields can be found in the operator product expansions of primaries and the nontrivial stress tensor components. Especially simple are the so called rational CFT's which involve only finite number of primary fields (and therefore contain finite spectrum of dimensions Δ_i).

Conceptually these two properties (which were in fact abstracted from explicit calculations in certain simple examples like free field theories) are enough to develop the whole structure as rich as the conformal field theory.

In 2D one can always choose locally an "isothermal" coordinate system where the metric is "conformally flat"

$$g_{ab}(x) = e^{\varphi(x)}\delta_{ab} \tag{2.4}$$

In the case of sphere such coordinates $(z, \bar{z})$ can be chosen almost globally, covering all the surface except for the "south pole" $z = \infty$. Simple transformation low (2.2) allows to express any correlation function of primary fields through the correlation functions on infinite flat plane with $g_{ab} = \delta_{ab}$ (again all fields are implied scalar)

$$\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle_\varphi = \prod_{i=1}^{n} e^{-\Delta_i\varphi(x_i)}\,\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle_{\varphi=0} \tag{2.5}$$

In this paper we'll concentrate on the metric of sphere of radius R. The Weil factor $\exp\varphi(z,\bar{z})$ in (2.4) can be chosen in the form

$$e^{\varphi(z,\bar{z})} = \frac{4R^2}{(1+z\bar{z})^2} \tag{2.6}$$

The scalar curvature is a positive constant

$$\widehat{R} = -4e^{-\varphi}\partial\bar{\partial}\varphi = 2R^{-2} \tag{2.7}$$

Metric (2.6) is invariant under the $O(3)$ group of coordinate transformations

$$z \to \frac{az+b}{\bar{a}-\bar{b}z} \tag{2.8}$$

where it is convenient to set $a\bar{a}+b\bar{b}=1$.

The stress tensor anomaly (2.1) readily prescribes the R dependence of the partition function

$$Z_{\rm CFT}(R) = R^{c/3} Z_0 \tag{2.9}$$

Here $Z_0 = Z_{\rm CFT}(1)$ is the CFT partition function at $R=1$. As far as I know, for this multiplier there are no theoretical predictions. In what follows I'll consider only the ratio $Z_{\rm CFT}(R)/Z_0$. A simple consequence of eq. (2.2) is the R dependence of any correlation function of primary fields on the sphere

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_R = R^{-2\sum_{i=1}^n \Delta_i}\,\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{R=1} \tag{2.10}$$

3. Perturbation

In non-trivial background (2.4) perturbed action (1.1) is replaced by

$$\mathcal{A}_\lambda = \mathcal{A}_{\rm CFT} + \frac{\lambda}{2\pi}\int \phi(x)e^{\varphi(x)}d^2x \tag{3.1}$$

where ϕ is again a relevant primary field of dimension $\Delta<1$.

Formal expression (3.1) gives rise to perturbative expansion in λ. Consider e.g., the perturbed partition function $Z_\lambda[\varphi]$. Formally

$$\frac{Z_\lambda[\varphi]}{Z_{\rm CFT}[\varphi]} = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(2\pi)^k k!}\int \langle\phi(x_1)\dots\phi(x_n)\rangle_\varphi \prod_{i=1}^n e^{\varphi(x_i)}d^2x_i \tag{3.2}$$

where the correlation functions are computed in unperturbed CFT but in non-trivial background φ. With the use of (2.5) one can "flatten" these correlation functions, reducing (3.2) to an expression which involves only the CFT correlation in trivial flat background $\varphi=0$

$$\frac{Z_\lambda[\varphi]}{Z_{\rm CFT}[\varphi]} = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(2\pi)^k k!}\int \langle\phi(x_1)\dots\phi(x_n)\rangle_{\rm CFT} \prod_{i=1}^n e^{(1-\Delta)\varphi(x_i)}d^2x_i \tag{3.3}$$

On the sphere (2.6) the R dependence is summarized as follows

$$\frac{Z_\lambda(R)}{Z_0 R^{c/3}} = z(h) \tag{3.4}$$

where the dimensionless combination

$$h = \lambda(2R)^{2-2\Delta} \tag{3.5}$$

is introduced. eq. (3.3) gives $z(h)$ as a regular power series in h

$$z(h) = \sum_{n=0}^{\infty}(-h)^n z_n \tag{3.6}$$

where explicitly

$$
\begin{aligned}
z_0 &= 1 \\
z_n &= \frac{1}{(2\pi)^n n!} \int \langle \phi(x_1) \ldots \phi(x_n) \rangle_{\rm CFT} \prod_{i=1}^{n} \frac{d^2 x_i}{(1 + x_i \bar{x}_i)^{2-2\Delta}} \qquad \text{at} \quad n > 0
\end{aligned} \tag{3.7}
$$

In CFT two- and three-point functions are fixed up to normalization constants. Assuming the usual in CFT normalization of primary fields we have

$$
\begin{aligned}
\langle \phi(x) \rangle_{\rm CFT} &= 0 \\
\langle \phi(x_1) \phi(x_2) \rangle_{\rm CFT} &= \frac{1}{|x_1 - x_2|^{4\Delta}} \\
\langle \phi(x_1) \phi(x_2) \phi(x_3) \rangle_{\rm CFT} &= \frac{C_{\phi\phi\phi}}{|(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^{2\Delta}}
\end{aligned} \tag{3.8}
$$

where $C_{\phi\phi\phi}$ is the corresponding structure constant, which is explicitly known in exact CFT constructions. First three coefficients in (3.6) are calculated in general form

$$
\begin{aligned}
z_1 &= 0 \\
z_2 &= \frac{1}{8(1 - 2\Delta)} \\
z_3 &= \frac{C_{\phi\phi\phi}}{48} \frac{\Gamma(2 - 3\Delta) \Gamma^3(1 - \Delta)}{\Gamma^3(2 - 2\Delta)}
\end{aligned} \tag{3.9}
$$

Higher perturbative integrals in (3.7) involve four and more point CFT correlation functions. These are not that universal as (3.8) and require more detailed information about a particular CFT model. The four-point calculation related to a particular model will be presented below.

If the series (3.6) is convergent, it defines the perturbed partition function as a function of R. In what follows we'll see an evidence that in certain cases this series is indeed convergent and, moreover, absolutely convergent, so that $z(h)$ is an entire function of h.

Anyway, the perturbative development (3.6) describes the $R \to 0$ asymptotic of the partition function $Z_\lambda(R)/Z_0$. Let's now turn to the opposite $R \to \infty$, or $h \to \infty$ limit.

4. Large R asymptotic

It seems intuitively clear that when $R \gg m^{-1}$ (where $m \sim \lambda^{1/(2-2\Delta)}$ is the typical mass scale in the perturbed model) the local physics is almost the same as in infinite flat space-time. In particular the leading exponential asymptotic of the partition function $Z_\lambda(R)$ is expected to be governed by the specific vacuum energy $\mathcal{E}_{\rm vac}$ in flat space

$$
\log Z_\lambda(R) \sim -4\pi R^2 \mathcal{E}_{\rm vac} + \cdots \tag{4.1}
$$

This vacuum energy is an important non-perturbative characteristic of a field theory model. For dimensional reasons

$$
\mathcal{E}_{\rm vac} = -A \lambda^{1/(1-\Delta)} \tag{4.2}
$$

where A is a dimensionless number. In integrable field theory this number is typically known exactly (see e.g., the example below).

To get an idea about further $R \to \infty$ corrections let's start with the relation

$$\frac{d \log Z_\lambda(R)}{dR^2} = -\langle \theta \rangle \tag{4.3}$$

and take the rescaled metric

$$e^{\varphi} = \left(1 + \frac{x\bar{x}}{4R^2}\right)^{-2} \tag{4.4}$$

As $R \to \infty$ this Weil factor is trivial $e^{\varphi} = 1$ and $\langle \theta \rangle$ coincides with the expectation value on infinite flat plane

$$\langle \theta \rangle_{\text{flat}} = 4\pi \mathcal{E}_{\text{vac}} \tag{4.5}$$

reproducing the leading asymptotic (4.1). For next corrections we should take into account the stress tensor anomaly (2.1)

$$\theta = -\frac{c}{6R^2} + \theta_{\text{flat}} \tag{4.6}$$

as well as the deviation of the metric from flat around the location of the operator $\theta(x)$. Let's put θ at $x = 0$ so that

$$e^{\varphi} = 1 - \frac{x\bar{x}}{2R^2} + \frac{3(x\bar{x})^2}{16R^4} + \cdots \tag{4.7}$$

General variation formula (X is any observable)

$$\delta \langle X \rangle = -\frac{1}{4\pi} \int \langle \theta(x) X \rangle \, e^{\varphi(x)} \delta\varphi(x) d^2x \tag{4.8}$$

to the first order in $1/R^2$ gives rise to the following R^{-2} correction to $\langle \theta \rangle$

$$\langle \theta(0) \rangle_{\text{sphere}} = \langle \theta(0) \rangle_{\text{flat}} + \frac{1}{8\pi R^2} \int \langle \theta(x)\theta(0) \rangle_{\text{flat}} \, x^2 d^2x - \frac{c}{6R^2} + O(R^{-4}) \tag{4.9}$$

The famous c-theorem sum rule [8] allows to evaluate the integral

$$\int \langle \theta(x)\theta(0) \rangle_{\text{flat}} \, x^2 d^2x = \frac{4\pi c}{3} \tag{4.10}$$

The anomalous R^{-2} term is cancelled out and the IR corrections to $\langle \theta \rangle$ appear at the order R^{-4}, so that

$$\langle \theta(0) \rangle_{\text{sphere}} \sim 4\pi \mathcal{E}_{\text{vac}} + \frac{b_1}{R^4} + \frac{2b_2}{R^6} + \cdots \tag{4.11}$$

with some (dimensional) coefficients b_1, $b_2, \ldots$ Thus, as it can be intuitively expected, at $R \to \infty$ there is no power correction to $Z_\lambda(R)$ and the IR expansion has the form

$$\log \frac{Z_\lambda(R)}{Z_0} \sim -4\pi R^2 \mathcal{E}_{\text{vac}} + \log Z_\infty + \frac{b_1}{R^2} + \frac{b_2}{R^4} + \cdots \tag{4.12}$$

Note that the sum rule (4.10) plays essential role in the cancellation of the $\log R$ correction. The (dimensional) integration constantZ_∞ scales as

$$Z_\infty = 2^{-c/3} \lambda^{-c/(6-6\Delta)} z_\infty \tag{4.13}$$

where z_∞ is a dimensionless number (numerical factor $2^{-c/3}$ is introduced for later convenience). Presently I don't understand neither physical meaning of this constant nor how it can be predicted on the basis of flat field theory. Nevertheless this parameter seems to be an important characteristic of the perturbed CFT on a sphere.

In terms of the variable (3.5) asymptotic (4.12) reads

$$\log z(h) = \pi A h^{1/(1-\Delta)} + \log z_\infty - \frac{c}{6-6\Delta}\log h + a_1 h^{-1/(1-\Delta)} + a_2 h^{-2/(1-\Delta)} + \cdots \quad (4.14)$$

where A is defined in (4.2) and $a_1 = 4b_1\lambda^{1/(1-\Delta)}$, $a_2 = 16b_2\lambda^{2/(1-\Delta)}$ etc.

Unlike z_∞, coefficients a_1, a_2, etc., (or $b_1, b_2, \ldots$ in (4.12)) can be expressed in terms of higher correlation functions of θ in flat space-time. For example

$$b_1 = \frac{1}{128\pi^2}\int \langle\theta(x_1)\theta(x_2)\theta(0)\rangle_{\text{flat}}\, x_1^2 x_2^2 d^2x_1 d^2x_2 - \frac{3}{64\pi}\int \langle\theta(x)\theta(0)\rangle_{\text{flat}}\, x^4 d^2x \quad (4.15)$$

In an integrable model the flat space correlation functions in principle can be constructed in terms of the form-factors of the operator θ, which are typically known exactly. However, even numerical calculation of the three-point function (or multipoint ones for the next coefficients) in this framework presents a separate problem.

Anyway, in the lack of any exact non-perturbative approach, a kind of "experimental" tool to measure the observables under consideration is very desirable.

5. Schrödinger picture in sphere geometry

In [3] the truncated conformal space (TCS) approach has been used to evaluate numerically certain characteristics of 2D perturbed CFT's. In that paper and a number of subsequent works [5] (see also [9] for most recent and practically important study) it was demonstrated that TCS is reasonably effective for many 2D models. In this section a way to apply similar approach in the spherical geometry is discussed.

For our purpose it is convenient to map the projective coordinates $(z, \bar{z})$ on the sphere to the "cylindric" ones (t, σ)

$$z = \exp\xi = \exp(t + i\sigma) \quad (5.1)$$

where the sphere metric (2.6) has the form

$$e^{\varphi(t,\sigma)} = \frac{R^2}{\cosh^2 t} \quad (5.2)$$

Consider the space of states on a circle of fixed σ. We will be interested in the "time" t evolution of these states and, in the TCS spirit, choose as the (time independent) basis the set of CFT states $|a\rangle$ (here index a runs over all the CFT states, primary and descendent). For simplicity we suppose that the basis is orthonormal. Any state (Schrödinger picture is implied where the states are time dependent)

$$|\Psi(t)\rangle = \sum_a \Psi^a(t)\,|a\rangle \quad (5.3)$$

is described thus by the wave function $\Psi^a(t)$. Forgetting for the moment the conformal anomaly (whose effect on the partition function is summarized to the multiplier $R^{c/3}$ in (2.9)) we consider the following time dependent hamiltonian

$$D(t) = D_0 + g(t)V \tag{5.4}$$

which generates translations in the t direction. Here

$$D_0 = L_0 + \bar{L}_0 \tag{5.5}$$

and L_0, $\bar{L}_0$ are the standard Virasoro generators acting on the CFT states. The interaction part of the evolution is constructed as

$$V = \frac{1}{2\pi}\int_0^{2\pi} \phi(0,\sigma)d\sigma \tag{5.6}$$

The time-dependent coupling constant $g(t)$ in (5.4) reads in terms of the "effective coupling" (3.5)

$$g(t) = -\frac{h}{(2\cosh t)^{2-2\Delta}} \tag{5.7}$$

Matrix elements $V\,|a\rangle = \sum_b B_a^b\,|b\rangle$ between CFT states are essentially the CFT structure constants

$$B_a^b = \frac{1}{2\pi}\int_0^{2\pi} \langle b|\,\phi(0,\sigma)\,|a\rangle\, d\sigma = C_{\phi\, a}^{b} \tag{5.8}$$

They are constructed explicitly in solvable CFT models. The t-evolution of the wave function is described by the time-dependent Schrödinger equation

$$-\frac{d}{dt}\,|\Psi(t)\rangle = D(t)\,|-\Psi(t)\rangle \tag{5.9}$$

In the CFT basis we arrive at the infinite dimensional system of linear differential equations

$$-\frac{d}{dt}\Psi^a(t) = \sum_b D_b^a(t)\Psi^b(t) \tag{5.10}$$

Different solutions to this system correspond to various operators placed at the points $t = -\infty$ and $t = \infty$ (i.e., at the "north" and "south" poles of the sphere). In particular, the solution $\Psi_{\text{vac}}^a(t)$ determined by the initial condition

$$\Psi_{\text{vac}}^a(-\infty) = \delta_I^a \tag{5.11}$$

where $a = I$ is the CFT state corresponding to the identity operator, describes the state radiated by smooth (no field) north pole. The reduced partition function (3.4) reads from this solution

$$z(h) = \lim_{t\to\infty} \Psi_{\text{vac}}^I(t) \tag{5.12}$$

It is easy to verify that the formal development (which implies that all infinite sums over intermediate states are convergent) of the solution Ψ_{vac} in powers of h results in the same perturbative series (3.6), (3.7). From other components of the wave function $\Psi^a_{\rm vac}(t)$ one can read-off the one-point functions at the south pole

$$(2R)^{2\Delta_a}\,\langle\Phi_a\rangle = \lim_{t\to\infty}\frac{\Psi^a_{\rm vac}(t)\exp(2\Delta_a t)}{\Psi^I_{\rm vac}(t)} \tag{5.13}$$

Here Φ_a is a scalar field corresponding to CFT state $|a\rangle$ and Δ_a is its dimension. Of course, this limit is in general divergent and requires standard UV renormalization, necessary to define the perturbed fields (see e.g. [10, 11]). This point will be discussed in more details in [12] while here we'll concentrate only on the partition function, where the limit (5.12) is well defined.

The TCS idea is quite simple: truncate the infinite dimensional CFT space of states up to certain maximal dimension and then treat the resulting finite-dimensional problem (5.10) numerically. Previous TCS experience shows that such procedure is often convergent (sometimes rather fast) with the increase of the truncation dimension.

6. Scaling Lee-Yang model

Scaling Lee-Yang model (SLYM) [13] is often used as a testing tool for various approximate approaches in 2D field theory. It is probably the simplest (apart from, of course, the free field theories) example of perturbed CFT. The model arises as a perturbation of the non-unitary CFT minimal model $\mathcal{M}(2/5)$. This CFT is rational and contains only two primary fields, the identity I of dimension 0 and the basic field $\phi = \Phi_{1,3}$ of dimension $\Delta = -1/5$, the central charge being negative $c = -22/5$. The basic operator product expansion reads

$$\phi(x)\phi(0) = (x\bar{x})^{2/5}I + (x\bar{x})^{1/5}C^{\phi}_{\phi\phi}\phi(0) + \cdots \tag{6.1}$$

where the three-ϕ structure constant is purely imaginary $C^{\phi}_{\phi\phi} = i\kappa$ and

$$\kappa = \left(\frac{\sqrt{5}-1}{2}\right)^{1/2}\frac{\Gamma^2(1/5)}{5\Gamma(3/5)\Gamma(4/5)} = 1.91131269990474\ldots \tag{6.2}$$

The petrurbed model

$$\mathcal{A}_{\rm SLYM} = \mathcal{A}_{\rm CFT} + \frac{i\lambda}{2\pi}\int\phi(x)e^{\varphi(x)}d^2x \tag{6.3}$$

is what is called the scaling Lee-Yang model.[1] In flat space-time ($\varphi \equiv 0$) it is integrable. Its particle content and factorized scattering theory are known exactly [14]. The spectrum contains only one massive particle of mass m.

Integrability allows to obtain many exact results about SLYM. E.g., the relation

$$m = k\lambda^{5/12}$$

[1] Note that the coupling λ in (6.3) is taken purely imaginary. This is nessessery to make the perturbed theory real.

between the mass m and the coupling constant λ is known [15]

$$k = \frac{4\sqrt{\pi}}{\Gamma(5/6)\Gamma(2/3)} \left[\frac{\Gamma(4/5)\Gamma(3/5)}{100\Gamma(1/5)\Gamma(2/5)}\right]^{5/24} = 1.2288903248\ldots \tag{6.4}$$

The bulk vacuum energy is also found exactly [2, 16]

$$\mathcal{E}_{\mathrm{vac}} = -\frac{m^2}{4\sqrt{3}} \tag{6.5}$$

so that the dimensionless parameter A in (4.2) is

$$A_{\mathrm{exact}} = \frac{k^2}{4\sqrt{3}} = 0.2179745\ldots \tag{6.6}$$

7. TCS at low truncations

In this section TCS for the Schrödinger equation (5.10) is applied to the scaling Lee-Yang model. This model is especially convenient for TCS thanks to the small number of primary fields and high degeneracy of modules at low levels. This results in low-dimensional truncated spaces even for relatively high truncation levels and makes the finite dimensional problem especially simple for numerical treatment. E.g., truncation level 5 (which means the maximum scale dimension in the truncated space $2\Delta_{\mathrm{max}} = 10$) gives 17-dimensional space of states.

Up to level 5 the matrix elements B_a^b can be found in [3]. Truncated system (5.10) was solved numerically and the reduced partition function $z(h)$ is evaluated as (5.12). In figure 1 the combination

$$\frac{Z_\lambda(R)}{Z_0} \exp(4\pi R^2 \mathcal{E}_{\mathrm{vac}}) = \Psi^I_{\mathrm{vac}}(\infty) R^{-22/15} \exp(4\pi R^2 \mathcal{E}_{\mathrm{vac}}) \tag{7.1}$$

is plotted as a function of dimensionless variable $mR = (k/2)h^{5/12}$, where m is the mass of SLYM particle. Data for different truncation levels are presented to illustrate the convergence.

First interesting feature we observe is a zero of the partition function at real positive $R = R_0$. Its position is estimated as

$$mR_0 = 0.88964\ldots \qquad \text{or} \qquad h_0 = 2.43083\ldots \tag{7.2}$$

At small R the data are well fitted by the perturbative series (3.6)

$$\frac{Z_\lambda(R)}{Z_0 R^{-22/15}} = z(h) = 1 - \frac{5}{56}h^2 - z_3 h^3 + \cdots \tag{7.3}$$

where z_3 reads, according to (3.9)

$$z_3 = \frac{\kappa}{48} \frac{\Gamma(13/5)\Gamma^3(6/5)}{\Gamma^3(12/5)} = 0.0229899\ldots \tag{7.4}$$

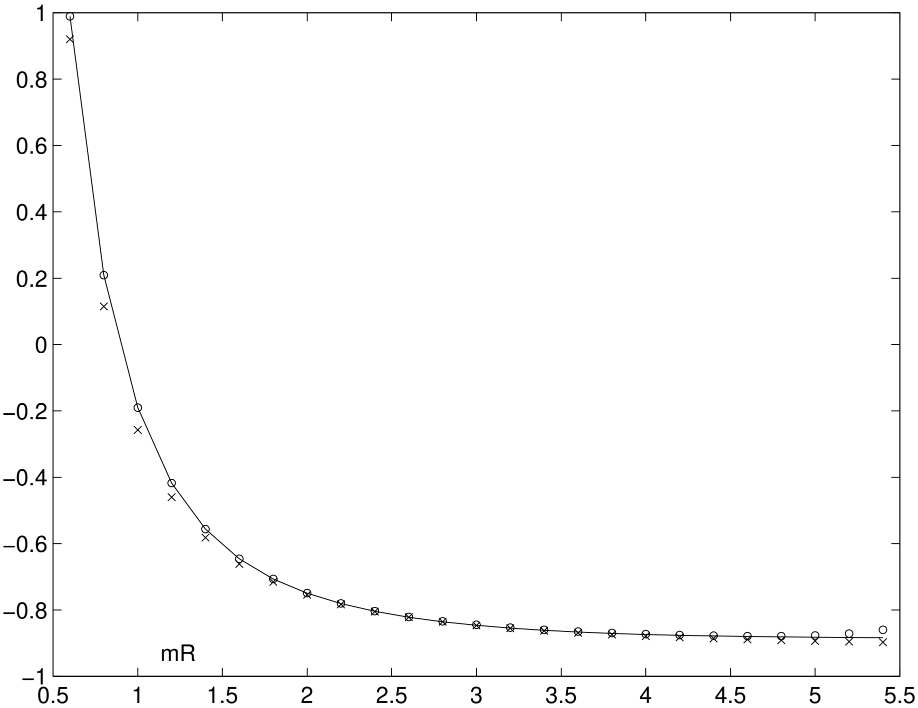

Figure 1: Combination (7.1) for truncation levels 5 (full curve) and 4 (small circles). The IR fit with formula (7.6) is plotted in crosses.

In the relatively stable interval $2.0 < mR < 3.0$ the level 5 data were fitted, according to (4.12), by the formula

$$\frac{Z_\lambda(R)\exp(4\pi R^2 \mathcal{E}_{\rm vac})}{Z_0} = Z_\infty \left(1 + \frac{b_1}{R^2}\right) \tag{7.5}$$

The best fit is achieved at

$$\begin{aligned} Z_\infty &= (-0.92 \pm 0.02) m^{-22/15} \\ b_1 &= (-0.72 \pm 0.2) m^{-2} \end{aligned} \tag{7.6}$$

For dimensionless constants z_∞ and a_1 in (4.14) this gives

$$\begin{aligned} z_\infty &= -0.45 \pm 0.01 \\ a_1 &= -1.9 \pm 0.5 \end{aligned} \tag{7.7}$$

These numbers enter the $h \to \infty$ asymptotic expansion

$$\log z(h) = \pi A h^{5/6} + \log z_\infty + \frac{11}{18}\log h + a_1 h^{-5/6} + a_2 h^{-10/6} + \cdots \tag{7.8}$$

where A is given by (6.6).

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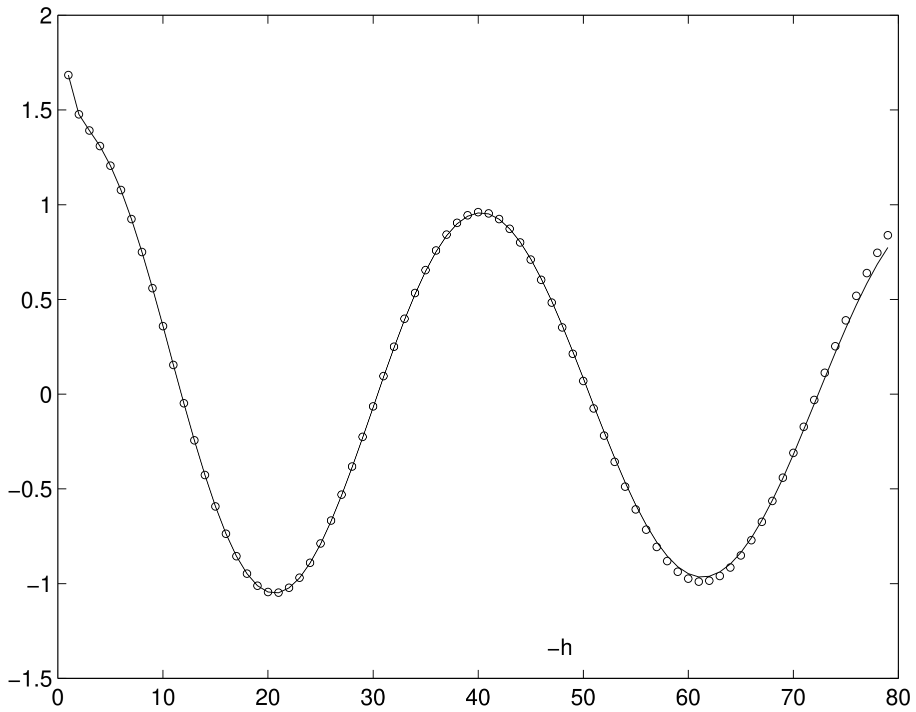

Figure 2: The negative h data for $z(h)\exp\left(\pi A\cos(\pi/6)(-h)^{5/6}\right)(-h)^{-11/18}$ at truncation levels 5 (continuous) and 4 (circles)

n	TCS	asymptotic	sum rules 1	sum rules 2	sum rules 3	sum rules 4
0	2.43083		2.45951	2.43099	2.43070	2.43069
1	-11.762	-12.41		-11.7162	-11.7732	-11.7650
2	-30.439	-30.66			-30.2335	-30.5363
3	-50.60	-51.05				-50.3603
4	-72.38	-72.94				
5		-95.98				
6		-120.0				
7		-144.9				

Table 1: Zeros of the partition function $z(h)$ estimated by TCS, leading asymptotic (8.1) and sum rules (8.3)

Analytic continuation for negative values of h is obtained by straightforward treatment of the truncated linear problem (5.10) with negative h. The negative h data show an oscillating pattern presented in figure 2, where the values of $z(h)\exp(\pi A\cos(\pi/6)(-h)^{5/6})$ $(-h)^{-11/18}$ are plotted against $-h$.

Numerical estimates for first few negative zeros h_n, $n = 1, 2, \ldots$ are gathered in the first column of table 1. It seems natural to expect infinite number of zeros on the negative h axis. Their positions are quite important for, as we'll argue below, $z(h)$ is entire function of h and therefore essentially determined by the location of its zeros.

8. Analytic considerations

For any truncated finite dimensional linear problem (5.10), (5.11) and (5.12) the resulting reduced partition function $z_{\text{trunc}}(h)$ is apparently an entire function of h. Let's suppose that this property holds also for the exact function $z(h)$. We admit also that all zeros of $z(h)$ are real, the first one h_0 being positive and the rest h_n, $n = 1, 2, \ldots$ negative (and accumulating at $h = -\infty$ along the real axis). This implies in particular that the asymptotic (7.8) holds in the whole complex h-plane as $|h| \to \infty$, apart from the negative real axis $\arg h = \pm\pi$. This analytic structure, combined with the perturbative information collected in (3.4), turns out to be quite restrictive.

Asymptotic (7.8) allows to estimate the leading behavior of the locations of zeros h_n at $n \to \infty$

$$-h_n = \left(\frac{2}{A}\left(n - \frac{1}{9}\right) + O(n^{-1})\right)^{6/5} \tag{8.1}$$

In table 1 this asymptotic estimate is compared with the approximate TCS data for first several zeros. The asymptotic does surprisingly good even for the first negative zero h_1.

With (8.1) the series

$$\frac{d}{dh} \log z(h) = \sum_{n=0}^{\infty} \frac{1}{h - h_n} \tag{8.2}$$

is convergent and defines the logarithmic derivative through the positions of zeros h_n. From (7.3) we have $\log z(h) = -\frac{5}{56}h^2 - z_3 h^3 + \ldots$ and therefore h_n satisfy the following sum-rules

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{h_n} = 0 \\ &\sum_{n=0}^{\infty} \frac{1}{h_n^2} = \frac{5}{28} \\ &\sum_{n=0}^{\infty} \frac{1}{h_n^3} = 3z_3 = 0.0689697\ldots \end{aligned} \tag{8.3}$$

A quick inspection of table 1 shows that the leading asymptotic (8.1) agrees the TCS data with at least 1% accuracy even for the forth zero h_3. Let's take this asymptotic expression for the exact positions of h_n at $n \geq N$ (N is certain number) and use the first N sum rules to recalculate the first N zeros. In practice, N is restricted by the number of available perturbative coefficients z_n. Numerical results are collected in table 1 for several values of N. The last column presents the numbers for $N = 4$ based on the numerical value of z_4 which will be evaluated separately in section 10. The numbers in the table illustrate good convergence of the solutions for first h_n with the increase of the perturbative order taken into account and generally are impressively close to the "experimental" positions measured by TCS. This numerical observation can be considered as a strong support of the suggested analytic structure of $z(h)$.

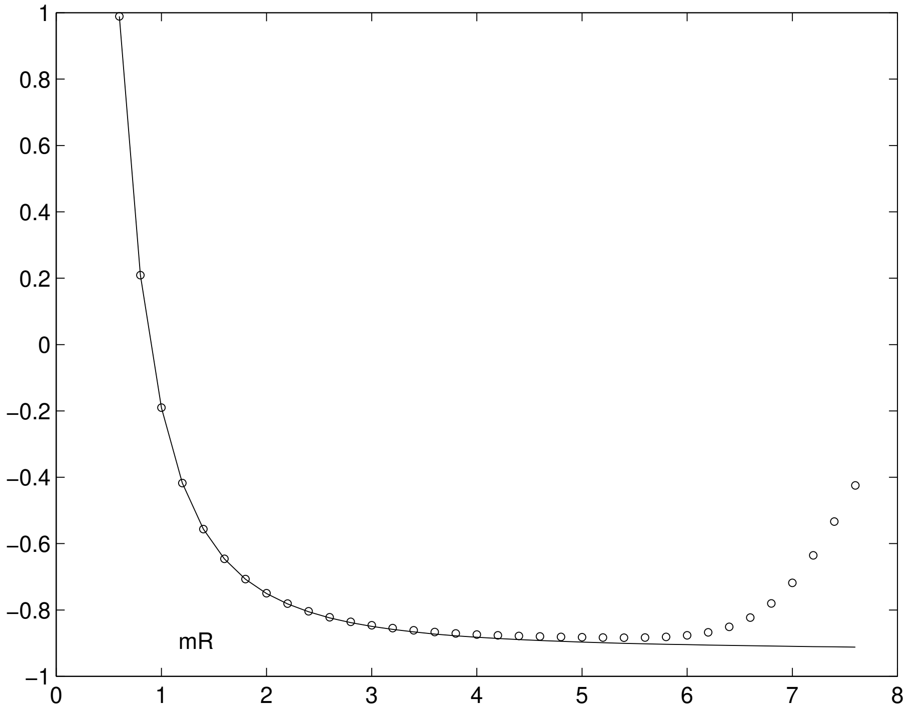

Figure 3: Weierstrass product (8.4) for $Z_\lambda(R)\exp(4\pi R^2 \mathcal{E}_{\rm vac})/Z_0$ based on the zeros in table 1 (full curve, $N = 3$ data from table 1 used for first 3 zeros) against the level 5 TCS data (circles).

Once all zeros are located with enough precision, the partition function can be restored as the convergent Weierstrass product

$$z(h) = \prod_{n=0}^{\infty} \left(1 - \frac{h}{h_n}\right) \exp\left(\frac{h}{h_n}\right) \tag{8.4}$$

(due to (8.1) and (8.3) the exponential multiplier in our case is not necessary and added here to improve the convergence). The resulting function $z(h)$ based on the above approximation for the zeros h_n is compared with level 5 TCS data for both positive and negative h in figures 3 and 4 respectively.

In principle one could improve the precision using next-to-leading correction to the asymptotic (8.1)

$$-h_n = \left(\frac{2}{A}\left(n - \frac{1}{9}\right) + \frac{a_1}{2\pi(n-1/9)} + O(n^{-2})\right)^{6/5} \tag{8.5}$$

with "experimental" value (7.7) for the coefficient a_1. Here I prefer not to go along this line. One reason is the low precision in the estimate (7.7). What is more important, the construction of $z(h)$ via (8.4) haven't used any numerical TCS data at all, the letter being only the basis for certain hypotheses about the analytic structure. Moreover, the only non-perturbative information (which in fact comes from the integrability of SLYM in flat space-time) entering our above calculations is the exact vacuum energy (6.6). In the next

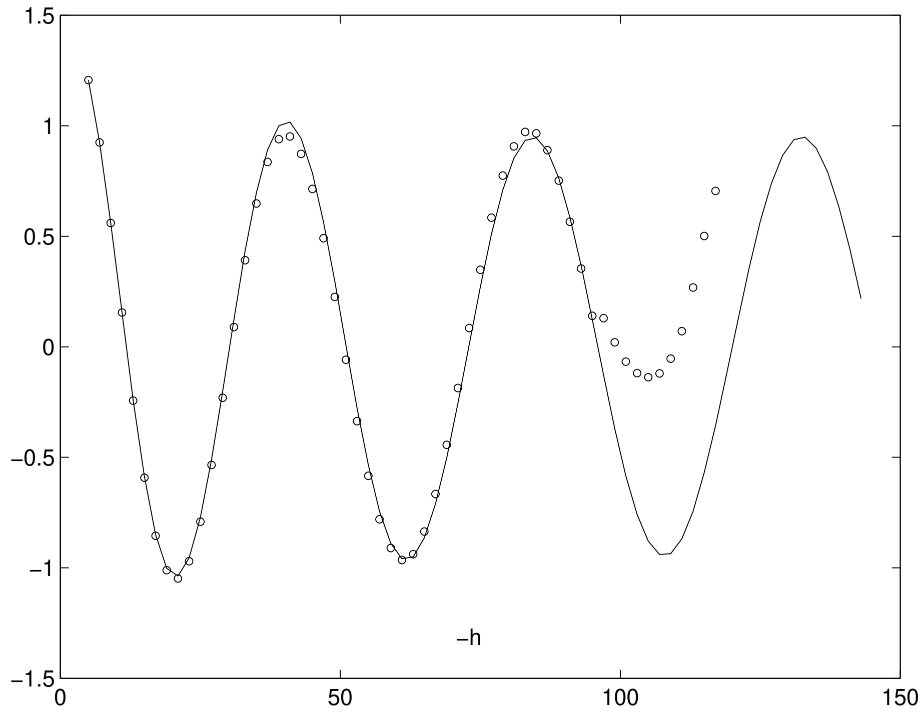

Figure 4: The same for negative values of the effective coupling constant h. Values of $z(h)\exp\left(\pi A\cos(\pi/6)(-h)^{5/6}\right)(-h)^{-11/18}$ are plotted.

section we'll see that even this income can be omitted (for the price of some tolerable loss of precision) and $z(h)$ is restored rather accurately on the only ground of few first perturbative coefficients.

9. Integrability off

Let's intentionally switch off the exact information (6.6) and take this number as an undeterminate parameter in the asymptotic expansion (7.8), together with z_∞ and a_1,$a_2, \ldots$ Of course, we continue to keep in mind the analytic structure of $z(h)$ described above and denote the zeros h_n, $n = 0, 1, 2, \ldots$ as before. In particular the large-N asymptotic (8.1) for h_n still holds, with now some unknown number A.

To handle the zeros it is convenient to define (at $\mathrm{Re}\, s > 1$) their zeta-function

$$\zeta_{\mathrm{LY}}(s) = \frac{e^{5i\pi s/6}}{h_0^{5s/6}} + \sum_{n=1}^{\infty} \frac{1}{(-h_n)^{5s/6}} \tag{9.1}$$

At $12/5 > \mathrm{Re}\, s > 1$ it can be computed from the partition function $z(h)$ as

$$\zeta_{\mathrm{LY}}(s) = \frac{\sin(5\pi s/6)}{\pi} \int_0^\infty h^{-5s/6} \frac{d\log z(h - i0)}{dh} dh \tag{9.2}$$

where the contour of integration is shifted to agree the branch chosen in (9.1). Vise versa

$$\log z(h) = \int_{-i\infty}^{i\infty} \frac{\pi \zeta_{\mathrm{LY}}(s) h^{5s/6}}{\sin(5\pi s/6)} \frac{ds}{2\pi i s} \tag{9.3}$$

where the integration contour goes to the left of the pole at $s = 6/5$ (in fact at $s = 6/5$ there is no pole because $\zeta_{\rm LY}(s)$ vanishes at this point due to the sum rules, so that the first pole in the right half-plane appears at $s = 12/5$) and to the right of the pole at $s = 1$. The following properties of $\zeta_{\rm LY}(s)$ are readily figured out:

1. $\zeta_{\rm LY}(-6n/5) = 0$ for $n = 1, 2, 3, \ldots$ This is to avoid wrong powers of h in the asymptotic expansion (7.8).

2. $\zeta_{\rm LY}(s)$ has simple pole at $s = 1$ with $\mathrm{res}_{s=1}\, \zeta_{\rm LY}(s) = A/2$. This provides the correct leading asymptotic in (7.8).

3. $\zeta_{\rm LY}(0) = 11/18$ to ensure correct $\log h$ contribution in the asymptotic, while the constant term requires

4. $\zeta'_{\rm LY}(0) = \dfrac{5}{6} \log z_\infty$

5. $\zeta_{\rm LY}(s)$ has simple poles at all integer negative $s = -n$, $n = 1, 2, \ldots$ and $\mathrm{res}_{s=-n}\, \zeta_{\rm LY}(s) = \pi^{-1} n \sin(5\pi n/6) a_n$ where $a_1, a_2, \ldots$ are the subleading coefficients in (7.8).

6. For all positive integer $k = 1, 2, \ldots$ we have by definition $\zeta_{\rm LY}(6k/5) = \sum_{n=0}^{\infty} (-h_n)^{-k}$, i.e., the numbers determined by the conformal perturbation theory. In particular (see (8.3))

$$\begin{aligned} \zeta_{\rm LY}\left(\frac{6}{5}\right) &= 0 \\ \zeta_{\rm LY}\left(\frac{12}{5}\right) &= 5/28 \\ \zeta_{\rm LY}\left(\frac{18}{5}\right) &= -3z_3 = -0.0689697\ldots \end{aligned} \tag{9.4}$$

The simplest approximation here is to replace all h_n for $n = 2, 3, \ldots$ by the leading asymptotic (8.1) while leaving A, h_0 and h_1 as unknowns. In other words we approximate $\zeta_{\rm LY}(s)$ as

$$\zeta^{(0)}_{\rm LY}(s) = \frac{e^{5i\pi s/6}}{h_0^{5s/6}} + \frac{1}{(-h_1)^{5s/6}} + \left(\frac{A}{2}\right)^s \zeta\left(s, \frac{17}{9}\right) \tag{9.5}$$

where $\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$ is the usual Riemann zeta-function. This expression identically satisfies property 3 while the sum rules (8.3) can be considered as a system of equations for undeterminate parameters A, h_0 and h_1. There are two real solutions. The first one

$$\begin{aligned} A &= 0.218156\ldots \\ h_0 &= 2.43068\ldots \\ h_1 &= -11.7540\ldots \end{aligned} \tag{9.6}$$

looks quite good when compared with the exact value (6.6) for A ((9.6) agrees (6.6) to 0.08%) and the "experimental" results for h_n from table 1. It's not clear if this remarkable precision is accidental. I'd like to remind that it's not a first instance of miraculous numerical success while applying different approximations to this particular perturbed CFT [3, 10].

With these data in hand we can estimate also the constant z_∞ evaluating $\zeta^{(0)\prime}_{\mathrm{LY}}(0)$

$$z_\infty = \frac{(2/A)^{5/3}}{h_0 h_1}\left(\frac{\Gamma(17/9)}{\sqrt{2\pi}}\right)^{6/5} = -0.4432\ldots \tag{9.7}$$

again in reasonable agreement with the TCS analysis.

Approximation (9.5) seems to be quite accurate in the right half-plane of s. However it has wrong analytic properties in the left half-plane. For example it has no pole at $s=-1$ (as well as at all negative integer s) and doesn't vanish at $s=-6n/5$, $n=1,2,\ldots$ E.g.,

$$\zeta^{(0)}_{\mathrm{LY}}\left(-\frac{6}{5}\right) = -h_0 - h_1 + \left(\frac{2}{A}\right)^{6/5}\zeta\left(-\frac{6}{5},\frac{17}{9}\right) = -3.28339\ldots \tag{9.8}$$

One can try to cure the situation near $s=-1$ introducing a pole at $s=-1$ by hand, i.e., taking say (this ansatz is obviously inspired by eq. (8.5))

$$\zeta^{(1)}_{\mathrm{LY}}(s) = \frac{e^{5i\pi s/6}}{h_0^{5s/6}} + \frac{1}{(-h_1)^{5s/6}} + \left(\frac{A}{2}\right)^{s}\zeta(s,17/9) - \frac{a_1}{2\pi}s\left(\frac{A}{2}\right)^{s+1}\zeta\left(s+2,\frac{17}{9}\right) \tag{9.9}$$

with a new undeterminate parameter a_1. Then, with the previous values of A, h_0 and h_1 we find that $\zeta^{(1)}_{\mathrm{LY}}(-6/5)=0$ if

$$a_1 \approx -2.060 \tag{9.10}$$

in tolerable agreement with (7.7). It can be verified that this correction doesn't violate essentially the sum rules (9.4) while z_∞ is slightly changed $z_\infty = -0.4566\ldots$

In principle one can continue the procedure and cure the value of $\zeta_{\mathrm{LY}}(-12/5)$ by adding more terms and trying to correct the analytic structure near $s=-2$, $s=-3$ etc. Due to the asymptotic character of the expansion (7.8), such iterative procedure is hardly convergent. Nevertheless the estimates of the unknown parameters A, h_0, h_1 and a_1 look quite reasonable.

The second real solution to (9.4) is numerically rather close to the first one

$$\begin{aligned} A &= 0.305435\ldots \\ h_0 &= 2.44244\ldots \\ h_1 &= -12.6153\ldots \end{aligned} \tag{9.11}$$

The estimate (9.7) gives now $z_\infty = -0.2345\ldots$ It remains a question whether this solution is an artefact of the approximate approach or we must take it seriously and look for a proper interpretation. Let me note that with the second value of A (9.11) the "jump" between the asymptotic estimate (8.1) of h_n and h_1 from (9.11) is relatively big. This makes questionable the whole reasoning above.

10. Four-point correction

The numerical success of the previous section makes it luring to improve the precision with higher-order perturbative terms. Here the forth order correction z_4 related to the four-point integral in (3.7) is evaluated numerically.

In general the CFT four point function

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{\mathrm{CFT}} = \frac{G_{\mathrm{CFT}}(y,\bar{y})}{|(x_1-x_4)(x_2-x_3)|^{4\Delta}} \tag{10.1}$$

depends essentially on the projective invariant

$$y = \frac{x_{12}x_{34}}{x_{14}x_{32}} \tag{10.2}$$

In order, the "reduced" four point function G_{CFT} satisfies the symmetries

$$G_{\mathrm{CFT}}(y,\bar{y}) = G_{\mathrm{CFT}}(1-y,1-\bar{y}) = |y|^{-4\Delta}\, G_{\mathrm{CFT}}\left(\frac{1}{y},\frac{1}{\bar{y}}\right) \tag{10.3}$$

and is usually expanded in terms of the four-point conformal blocks

$$G_{\mathrm{CFT}}(y,\bar{y}) = \sum_i C^i_{\phi\phi} C_{i\phi\phi} \mathcal{F}(\Delta,\Delta_i,y)\mathcal{F}(\Delta,\bar{\Delta}_i,\bar{y}) \tag{10.4}$$

where the sum is over the set of primary fields Φ_i (with dimensions $(\Delta_i,\bar{\Delta}_i)$) entering the operator product expansion of two fields ϕ, $C^i_{\phi\phi}$ being the corresponding structure constants. Functions $\mathcal{F}(\Delta,\Delta_i,y)$ are the standard four-point conformal blocks. In particular, in the Lee-Yang CFT the sum contains only two terms related to the identity operator I and the basic field ϕ

$$G_{\mathrm{LY}}(y,\bar{y}) = |\mathcal{F}_I(y)|^2 - \kappa^2\,|\mathcal{F}_\phi(y)|^2 \tag{10.5}$$

and the blocks can be expressed in terms of hypergeometric functions

$$\begin{aligned}\mathcal{F}_I(y) &= y^{2/5}(1-y)^{1/5} F\left(\frac{2}{5},\frac{3}{5},\frac{6}{5},y\right)\\ \mathcal{F}_\phi(y) &= y^{1/5}(1-y)^{1/5} F\left(\frac{1}{5},\frac{2}{5},\frac{4}{5},y\right)\end{aligned} \tag{10.6}$$

so that the four point function is known explicitly.

To evaluate the integral (3.7) it is convenient to fix the projective invariant (10.2) and perform first the remaining three integrations. This introduces the "classical Liouville four-point function"

$$\begin{aligned}G_L(y,\bar{y}) &= \int \frac{1}{|x_{14}x_{23}|^{4-4\sigma}}\delta^{(2)}\left(y-\frac{x_{12}x_{34}}{x_{14}x_{32}}\right)\prod_{i=1}^{4}\frac{d^2x_i}{(1+x_i\bar{x}_i)^{2\sigma}}\\ &= \int \frac{\left(|yx_{32}-x_{34}|^2 + |x_{32}x_4y - x_2x_{34}|^2\right)^{-2\sigma}}{|x_{23}x_{42}x_{34}|^{2-4\sigma}}\prod_{i=2}^{4}\frac{d^2x_i}{(1+x_i\bar{x}_i)^{2\sigma}}\end{aligned} \tag{10.7}$$

where $\sigma = 1 - \Delta$. It is easy to check that

$$G_L(y, \bar{y}) = G_L(1-y, 1-\bar{y}) = |y|^{-4\sigma} G_L\left(\frac{1}{y}, \frac{1}{\bar{y}}\right) \tag{10.8}$$

Integral (10.7) can be reduced to a single one with the hypergeometric function

$$G_L(y, \bar{y}) = \frac{\pi^3}{4\sigma - 1} \int_0^1 \frac{dx}{x(1-x)} F\left(2\sigma, 2\sigma, 4\sigma, 1 - \frac{|y|^2}{x} - \frac{|1-y|^2}{1-x}\right) \tag{10.9}$$

Another fast-convergent integral representation for G_L, similar to (10.4), is useful for numerical calculations

$$G_L(y) = \pi^3 \int_C \frac{(2s-1)ds}{2\pi i} \frac{\Gamma^2(2\sigma+s-1)\Gamma^2(2\sigma-s)\Gamma^4(s)}{\Gamma^4(2\sigma)\Gamma^2(2s)} \mathcal{F}^{(\mathrm{cl})}(\sigma, s, y) \mathcal{F}^{(\mathrm{cl})}(\sigma, s, \bar{y}) \tag{10.10}$$

Here the "classical block" $\mathcal{F}^{(\mathrm{cl})}(\sigma, s, y)$ is expressed again through the hypergeometric function

$$\mathcal{F}^{(\mathrm{cl})}(\sigma, s, y) = y^{s-2\sigma} F(s, s, 2s, y) \tag{10.11}$$

and the integration contour C goes up along the imaginary axis to the left from the poles of $\Gamma^2(2\sigma - s)$ and to the right from the rest singularities of the integrand.

The four-point integral now reads

$$z_4 = \frac{1}{64\pi} \int_D G_{\mathrm{LY}}(y, \bar{y}) G_L(y, \bar{y}) d^2 y \tag{10.12}$$

where, due to symmetries (10.3) and (10.8) the integration can be restricted within the segment $D = \{-\sqrt{1 - (\mathrm{Im}\,\xi)^2} < \mathrm{Re}\,\xi < 1/2\}$ (the corresponding factor 6 is already taken into account in (10.12)). Numerical evaluation of (10.12) with (10.5) gives

$$z_4 = -0.00318918\ldots \tag{10.13}$$

Now, with this coefficient known, we can proceed further along the line of the previous section: approximate zeros at $n > 2$ by the asymptotic formula (8.1) with A to be determined self consistently from the first four sum rules (8.3) together with the first three zeros. Suitable numerical solution (there are now three solutions, the most reasonable is quoted) reads

$$\begin{aligned} A &= 0.218036\ldots \\ h_0 &= 2.43069\ldots \\ h_1 &= -11.7667\ldots \\ h_2 &= -30.3670\ldots \end{aligned} \tag{10.14}$$

In this solution A approximates the exact value (6.6) to better then 0.03% precision. Similarly to (9.7) one finds also the improved estimate for the constant z_∞

$$z_\infty = -\frac{(2/A)^{43/15}}{h_0 h_1 h_2} \left(\frac{\Gamma(26/9)}{\sqrt{2\pi}}\right)^{6/5} = -0.4473\ldots \tag{10.15}$$

It's expected that further perturbative information could give even better numerical information about the details of the partition function. It should be noted, however, that the numerical-analytic methods like used above require the input coefficients z_n to a very high numerical precision. This makes the numerical problem at higher orders rather involved. It particular, simple Monte-Carlo estimates of the higher perturbative integrals (3.7) prove useless.

11. Remarks

- The most interesting conclusion we're led to by both TCS experiment and analytic considerations (in the framework of SLYM) is that the spherical partition function of a perturbed CFT may have quite specific analytic properties in the coupling constant (3.5). Are these properties are typical for any perturbed CFT on a sphere? Exactly solvable example of free massive fermion (see appendix A, where the partition functions of free massive fields are presented explicitly) shows very similar analytic structure. However, the free massive boson (appendix A.2) gives an immediate counterexample. Analytic structure is more complicated, the partition function developing an infinite sequence of brunch points etc. Probably the simple analytic picture is characteristic for rational perturbed CFT's. It remains to be answered if integrability of the corresponding model in flat plays any role in this analytic structure.

- Consider the derivative $z'(h) = dz(h)/dh$, which is again supposed to be an entire function of h with the asymptotic at $|z| \to \infty$, $-\pi < \arg z < \pi$

$$z'(h) \sim \exp\left(\pi A h^{5/6} + \frac{4}{9}\log h + \text{const} + \cdots\right) \tag{11.1}$$

Denote its zeros h'_n. Perturbative expansion (7.3) gives exactly the first zero $h'_0 = 0$. From (11.1) the large-N asymptotic of h'_n is read off

$$-h'_n = \left(\frac{2}{A}\left(n + \frac{1}{18}\right) - +O(n^{-1})\right)^{6/5} \tag{11.2}$$

In particular, the convergent product

$$z'(h) = -\frac{5}{28} h \prod_{n=1}^{\infty}\left(1 - \frac{h}{h'_n}\right) \tag{11.3}$$

relates $z'(h)$ to the positions of its zeros. The third term in the series (7.3) results in the sum rule for h'_n

$$\sum_{n=1}^{\infty}\frac{1}{h'_n} = -\frac{84}{5} z_3 \tag{11.4}$$

Take the leading asymptotic (11.2) as exact for $n \geq 1$. The sum rule becomes

$$\left(\frac{A}{2}\right)^{6/5}\sum_{n=1}^{\infty}\frac{1}{(n+1/18)^{6/5}} = \frac{84}{5} z_3 = 0.3862303 \tag{11.5}$$

and gives for A

$$A = 0.218767... \tag{11.6}$$

This accurate result is again remarkable, especially in view of the amazingly simple way it is obtained. Also, unlike more complicated system in section9, this way gives unique solution.

- Note, that the derivative $z'(h)$ is related to the (unnormalized) one-point function of the perturbing operator ϕ

$$z(h)(2R)^{-2/5} \langle \phi \rangle = -2z'(h) \tag{11.7}$$

In the TCS approach

$$z(h)(2R)^{-2/5} \langle \phi \rangle = \lim_{t\to\infty} \Psi^{\phi}_{\text{vac}}(t) \exp\left(-\frac{2t}{5}\right) \tag{11.8}$$

(no subtractions are required for this one-point function). Apparently, for any finite-dimensional TCS problem this is again an entire function of h. It seems natural to generalize the observed analytic structure for arbitrary unnormalized one-point function

$$z(h)(2R)^{2\Delta_a} \langle \Phi^a \rangle = \lim_{t\to\infty} \left(\Psi^{a}_{\text{vac}}(t) \exp(2\Delta_a t) - \text{subtractions}\right) \tag{11.9}$$

and then use it to get more information on the grounds of perturbation theory and analytic properties. These attempts will be reported in a separate publication [12].

- It would be interesting to compare both the experimental (7.6) and "theoretical" (9.10) estimates of the first subleading IR correction to the partition function with the prediction (4.15) evaluated with the use of exact form-factors of θ [10].

- TCS approach as described above proves to provide reasonable experimental data for the perturbed CFT on the sphere. It is well known however, that this method works well mainly for perturbed rational CFT's, and moreover, when the conformal perturbative integrals are convergent. In particular, many interesting models where the dimension of the perturbing operator is close to 1, or with asymptotically free marginal perturbation are completely unaccessible by this numerical scheme. A suitable approach which would allow to overcome this restriction still remains to be developed.

- Nevertheless, there are still many perturbed CFT's (like the sin-Gordon model with sufficiently small β) where TCS must perform reasonably and the hypotheses proposed in this article can be checked against the experimental data. Work in this direction is in progress.

A. Free fields on a sphere

A.1 Majorana fermion

Free massive Majorana fermion in curved geometry with conformally flat metric (2.4) is defined by the action

$$\begin{aligned}\mathcal{A}_{\text{ferm}} &= \frac{1}{\pi}\int d^2x\left[\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi}+ime^{\varphi/2}\bar{\psi}\psi\right]\\ &= \frac{1}{\pi}\int d^2x\left[\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi}\right]+\frac{m}{2\pi}\int\varepsilon(x)e^{\varphi(x)}d^2x\end{aligned}\tag{A.1}$$

where m is the mass of the fermion, which is just the coupling constant in the framework of perturbed CFT, while the energy density

$$\varepsilon = 2ie^{-\varphi/2}\bar{\psi}\psi\tag{A.2}$$

plays the role of perturbing operator. This action leads to the Dirac equations of motion

$$\begin{aligned}2\bar{\partial}\psi &= ime^{\varphi/2}\bar{\psi}\\ 2\partial\bar{\psi} &= -ime^{\varphi/2}\psi\end{aligned}\tag{A.3}$$

We're interested in the partition function $Z_m^{(\mathrm{f})}(R)$ of this field theory on the sphere of radius R. Obviously its derivative in m is determined by the expectation value of ε. By symmetry $\langle\varepsilon(x)\rangle$ is independent on x and

$$\frac{d\log Z_m^{(\mathrm{f})}(R)}{dm} = -2R^2\left\langle\varepsilon\right\rangle\tag{A.4}$$

To evaluate $\langle\varepsilon\rangle$ consider the two-point functions of the fermionic fields. For the symmetry reasons

$$\begin{aligned}\langle\psi(z,\bar{z})\psi(0)\rangle &= \frac{1}{z}f(z\bar{z})\\ \left\langle\bar{\psi}(z,\bar{z})\psi(0)\right\rangle &= -ig(z\bar{z})\end{aligned}\tag{A.5}$$

Equations of motion (A.3) result in the following simple system

$$\begin{aligned}(1-t)f'(t) &= rg(t)\\ tg'(t) &= rf(t)\end{aligned}\tag{A.6}$$

where $r = mR$ and

$$t = \frac{z\bar{z}}{(1+z\bar{z})} = \sin^2\left(\frac{s}{2R}\right)\tag{A.7}$$

is related to the geodesic distance s between 0 and z. Suitable solution, which is regular at $t = 1$ (i.e., at the south pole of the sphere $z = \infty$) and properly normalized at $t = 0$ reads

$$\begin{aligned}f(z\bar{z}) &= \frac{\Gamma(1+ir)\Gamma(1-ir)}{2}F\left(ir,-ir,1,(1+z\bar{z})^{-1}\right)\\ g(z\bar{z}) &= -\frac{r\Gamma(1+ir)\Gamma(1-ir)}{2(1+z\bar{z})}F\left(1+ir,1-ir,2,(1+z\bar{z})^{-1}\right)\end{aligned}\tag{A.8}$$

Taking the limit $z\bar{z} \to 0$ one can read-off the expectation value $\langle \varepsilon(0) \rangle$ at the north pole $z = 0$

$$\begin{aligned}\langle \varepsilon \rangle &= 2g\left(\epsilon^2 e^{-\varphi(0)}\right) e^{-\varphi(0)/2} \\ &= \frac{m}{2}\left(\log\frac{\epsilon^2}{4R^2} + \psi(1+ir) + \psi(1-ir) - 2\psi(1)\right)\end{aligned} \tag{A.9}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and ϵ is the invariant UV cutoff. Integrating now (A.4) and taking into account the conformal anomaly (2.9) one finds

$$\frac{Z_m^{(\mathrm{f})}(R)}{Z_0^{(\mathrm{f})} R^{1/6}} = \exp\left(-r^2\left(\log\frac{\epsilon}{2R} + C + 1\right)\right)\frac{\exp(2\zeta'(-1))}{\Gamma_2(1+ir|1,1)\Gamma_2(1-ir|1,1)} \tag{A.10}$$

where C is the Euler's constant and $\Gamma_2(x|1,1)$ is a particular case of the Barnes double gamma function [17]

$$\log\Gamma_2(x|\omega_1,\omega_2) = \frac{d}{ds}\sum_{m,n=0}^{\infty}(x + m\omega_1 + n\omega_2)^{-s}\Bigg|_{s=0} \tag{A.11}$$

Ferimionic partition function $Z_m^{(\mathrm{f})}(R)$ is an entire function of r. It has zeros at $r = \pm i$, $\pm 2i$, $\pm 3i \ldots$ of multiplicities 1, 2, 3... respectively, which are of course related to the zero modes of the spherical Dirac operator at these values of mR.

At small R this partition function develops as follows

$$\log\frac{Z_m^{(\mathrm{f})}(R)}{Z_0^{(\mathrm{f})}} = \frac{1}{6}\log R - r^2\log\frac{\epsilon}{2R} + \sum_{k=1}^{\infty}\frac{(-)^k\zeta(2k+1)}{k+1}r^{2k+2} \tag{A.12}$$

The large R behavior is determined by the asymptotic expansion

$$\frac{Z_m^{(\mathrm{f})}(R)}{Z_0^{(\mathrm{f})}} \sim m^{-1/6} z_\infty^{(\mathrm{f})}\exp\left(-4\pi R^2\mathcal{E}_{\mathrm{vac}}^{(\mathrm{f})} + \frac{1}{120m^2R^2} + \frac{1}{504m^4R^4}\cdots\right) \tag{A.13}$$

where

$$\mathcal{E}_{\mathrm{vac}}^{(\mathrm{f})} = \frac{m^2}{4\pi}\left(\log\frac{m\epsilon}{2} + C - \frac{1}{2}\right) \tag{A.14}$$

is the bulk vacuum energy of the free fermion in flat space-time and

$$z_\infty^{(\mathrm{f})} = \exp(2\zeta'(-1)) = 0.718318\ldots \tag{A.15}$$

A.2 Massive boson

The action of free massive boson ϕ in curved geometry reads

$$\mathcal{A}_{\mathrm{boson}} = \frac{1}{4\pi}\int d^2x\left[(\partial_a\phi)^2 + m^2 e^{\varphi}\phi^2\right] \tag{A.16}$$

and leads to the free field equation of motion

$$4\partial\bar{\partial}\phi = m^2 e^{\varphi}\phi \tag{A.17}$$

Again, the derivative in m^2 of the bosonic partition function $Z_m^{(\mathrm{b})}(R)$ is expressed in terms of $\langle\phi^2\rangle$ on the sphere

$$\frac{d\log Z_m^{(\mathrm{b})}(R)}{dm^2} = -\frac{1}{4\pi}\int\langle\phi^2\rangle\, e^{\varphi} d^2x = -R^2\langle\phi^2\rangle \tag{A.18}$$

From (A.17) it follows that the two-point function $G(z\bar z) = \langle\phi(z,\bar z)\phi(0)\rangle$ satisfies hypergeometric differential equation

$$t(1-t)G_{tt} + (1-2t)G_t - m^2R^2G = 0 \tag{A.19}$$

where t is the same as in (A.7). Relevant solution, regular at the "south pole" $z=\infty$ and properly normalized at $z\to 0$, has the form

$$G(z\bar z) = \frac{\Gamma(1/2+is)\Gamma(1/2-is)}{2} F\left(\frac{1}{2}+is, \frac{1}{2}-is, 1, (1+z\bar z)^{-1}\right) \tag{A.20}$$

where

$$s = \left(m^2R^2 - 1/4\right)^{1/2} \tag{A.21}$$

Considering the subleading term in $z\to 0$ asymptotic of this function we find

$$\begin{aligned}\langle\phi^2\rangle &= G\left(\epsilon^2 e^{-\varphi(0)}\right)\\ &= -\frac{1}{2}\left(\log\frac{\epsilon^2}{4R^2} + \psi\left(\frac{1}{2}+is\right) + \psi\left(\frac{1}{2}-is\right) - 2\psi(1)\right)\end{aligned} \tag{A.22}$$

where again the UV cutoff ϵ appears explicitly.

Integration of (A.18) is not that straightforward as in the fermionic case. Expectation value (A.22) is singular at $s=\pm i/2$, i.e., at $m=0$. This is of course a manifestation of the zero mode of (A.17) at $m=0$. It leads to an extra multiplier $(mR)^{-1}$ (in addition to the conformal anomaly (2.9)) in the small R behavior of the partition function. Normalization of $Z_m^{(\mathrm{b})}(R)$ becomes ambiguous even if $Z_0^{(\mathrm{b})}$ is fixed in some way. Below we adopt the normalization where $Z_m^{(\mathrm{b})}(R)/Z_0^{(\mathrm{b})} \sim R^{1/3}/(mR)$ at $R\to 0$. With this convention

$$\begin{aligned}&\frac{Z_m^{(\mathrm{b})}(R)}{Z_0^{(\mathrm{b})}R^{1/3}} = \\ &\exp\left(m^2R^2\left(\log\frac{\epsilon}{2R} + C + 1\right)\right)\frac{\exp(-2\zeta'(-1))\Gamma_2(3/2+is|1,1)\Gamma_2(3/2-is|1,1)}{(2\cosh(\pi s))^{1/2}}\end{aligned} \tag{A.23}$$

Bosonic partition function has brunch points of order $-1/2$ at $s=\pm i/2$, of order $-3/2$ at $s=\pm 3i/2$, of order $-5/2$ at $s=\pm 5i/2$ etc. These singularities are caused by the zero modes of (A.17). In terms of the variable m^2R^2 they turn to similar brunch points at $m^2R^2 = -n(n+1)$, $n=0,1,2,\ldots$

Small R behavior reads

$$\frac{Z_m^{(\mathrm{b})}(R)}{Z_0^{(\mathrm{b})}} \sim \frac{R^{1/3}}{mR}\exp\left(\frac{m^2R^2}{2}\left(\log\frac{\epsilon^2}{4R^2}+1\right) + \frac{1}{4}m^4R^4 - \frac{\zeta(3)-1}{3}m^6R^6 + \cdots\right) \tag{A.24}$$

while at $R \to \infty$

$$\frac{Z_m^{(\mathrm{b})}(R)}{Z_0^{(\mathrm{b})}} = m^{-1/3} z_\infty^{(\mathrm{b})} \exp\left(-4\pi R^2 \mathcal{E}_{\mathrm{vac}}^{(\mathrm{b})} + \frac{1}{30m^2R^2} + \frac{2}{315m^4R^4} \cdots\right) \tag{A.25}$$

where

$$\mathcal{E}_{\mathrm{vac}}^{(\mathrm{b})} = -\frac{m^2}{4\pi}\left(\log\frac{m\epsilon}{2} + C - \frac{1}{2}\right) \tag{A.26}$$

is the bulk vacuum energy of the massive free boson in flat, and

$$z_\infty^{(\mathrm{b})} = \exp\left(\frac{1}{4} - 2\zeta'(-1)\right) = 1.78754\ldots \tag{A.27}$$

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